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Chaos, metastability and partial synchronization in systems with long-range interactions

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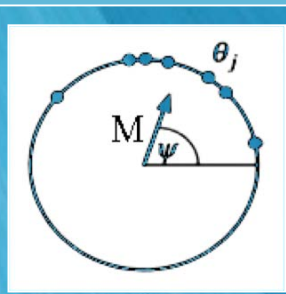
SYSTEM OF N DRIVEN AND DAMPED FULLY COUPLED OSCILLATORS

$$\ddot{\vartheta}_i + B\dot{\theta}_i + M\,K\sin(\theta_i - \Psi) = \omega_i \qquad i = 1,\dots,N \qquad \theta_i = phases$$

B=damping coefficient; K=coupling strength; ω_i=driven frequencies; Ψ=global phase

Mean Field Order Parameter

$$\vec{M} = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j} = M\,e^{i\Psi}$$



Conservative case

$$B = 0, K = 1, \omega_i = 0$$

$$\ddot{\vartheta}_i = M\sin(\Psi - \theta_i) \qquad i = 1,\dots,N$$

HMF Model

Antoni and Ruffo PRE 52 (1995) 2361

Dissipative case

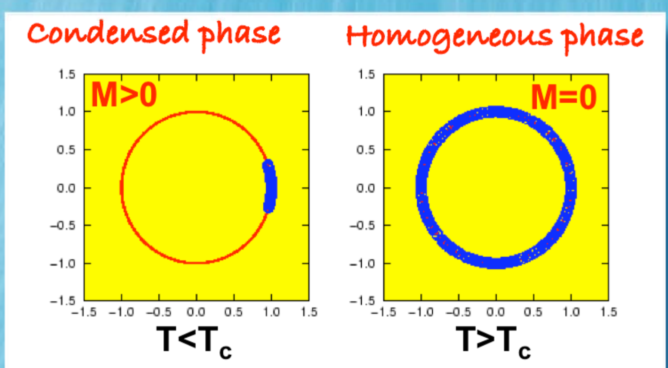
$$B \gg 1, K \geq 0, \omega_i \neq 0$$

$$\dot{\theta}_i = \omega_i + M\,K\sin(\Psi - \theta_i) \qquad i = 1,\dots,N$$

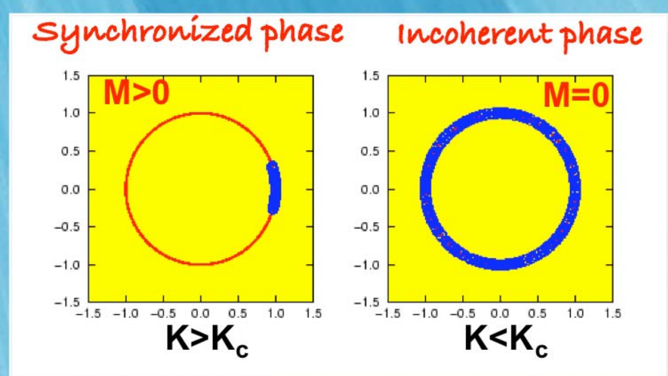
Kuramoto Model

Y. Kuramoto, Chemical Oscillations, Waves, and Turbulence, Springer, Berlin, 1984

Both the models show a phase transition from a condensed (synchronized) phase to an homogeneous (incoherent) one above a given threshold of a control parameter (Temperature for the HMF model, coupling strength K for the Kuramoto model):



The oscillators (rotors) can be usefully viewed as particles rotating without collisions on a unit circle



Both the models are representative of large classes of interesting physical systems showing metastability, synchronization and phase transitions:

- Out-of-equilibrium complex sytems
- Nuclear multifragmentation
- Atomic clusters
- Astrophysical systems
- Plasma physics
- Social and biological systems (fireflies synchronization, pacemaker cells, …)
- Josephson's Junctions
- Laser arrays
- Landau damping in plasma physics

Stationarity and Equilibrium Behavior

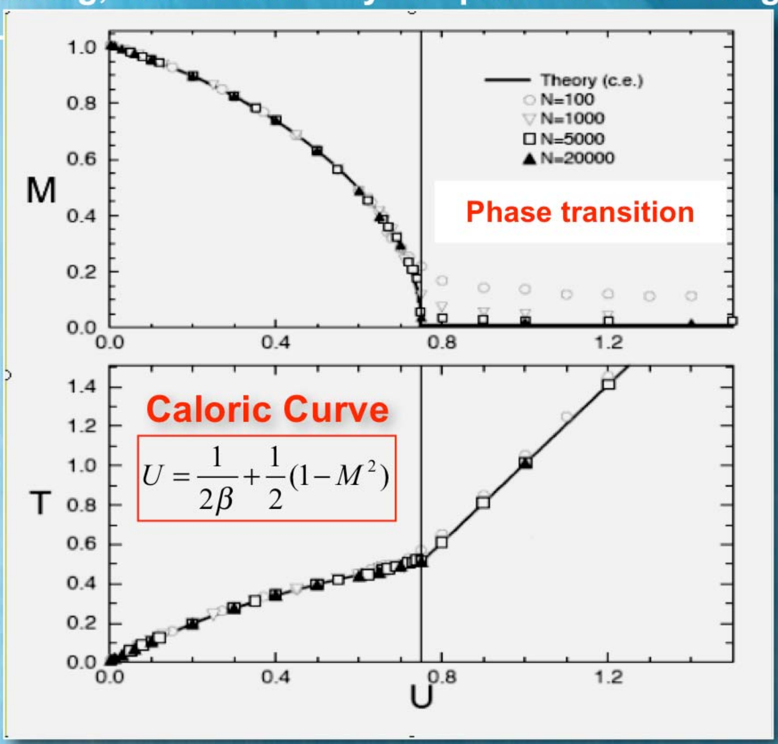
HMF Model of XY inertial rotators

$$H = K + V = \sum_{i=1}^{N} \frac{p_i^2}{2} + \frac{1}{2N} \sum_{i,j=1}^{N} \left[1 - \cos(\theta_i - \theta_j) \right] \qquad i = 1, \ldots, N$$

$$p_i = \dot{\theta}_i = conjugate\ velocities \qquad T = \beta^{-1} = \frac{2K}{N} (temperature)$$

$$U = H / N \quad (energy\ density)$$

The Hamiltonian Mean Field model is exactly solvable with a mean field approach in both Canonical and Microcanonical ensembles, showing a second order phase transition at a critical temperature T_c=0.5 and at a critical energy density U_c=0.75. The predictions at equilibrium are in good agreement with Microcanonical molecular dynamics simulations. Near the phase transition the system is characterized by strong chaos and mixing, as shown by a peak in the Largest L



Magnetization and energy density at the critical point

$$M \approx \frac{4}{\beta} \sqrt{\frac{1}{2} - \frac{1}{\beta}} \sim (T_C - T)^{\frac{1}{2}}$$

$$U \approx \frac{1}{2\beta} \left(1 - \frac{8(\beta - 2)}{\beta^2} \right) + \frac{1}{2}$$

Largest Lyapunov Exponent:

$$\lambda_1 = \lim_{t \to \infty} \frac{1}{t} \ln \frac{d(t)}{d(0)}$$

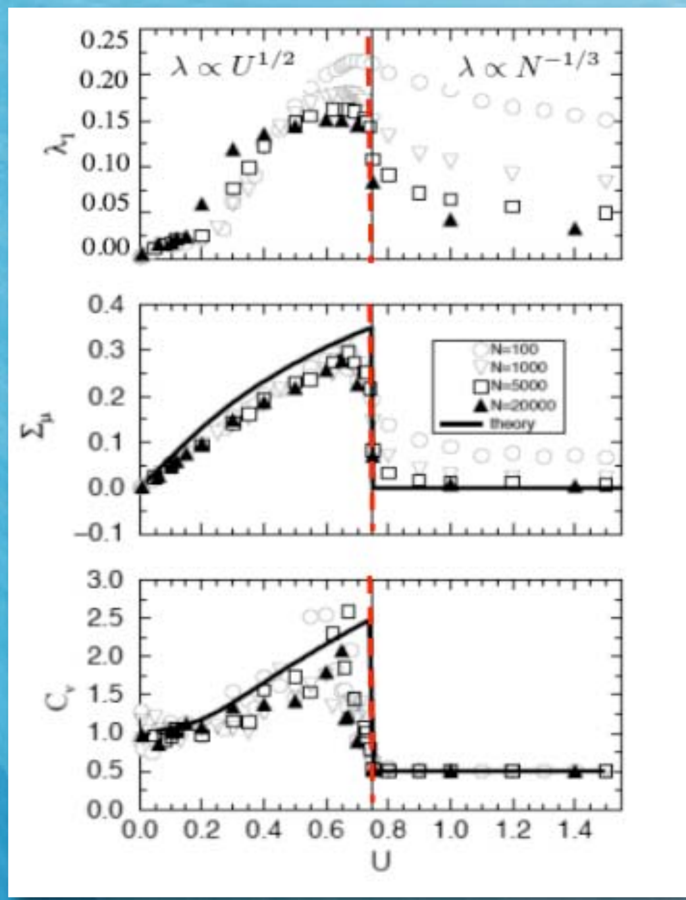
$$d(t) = \sqrt{\sum_i \left[(d\theta_i)^2 + (dp_i)^2 \right]}$$

Kinetic Energy fluctuations

$$\Sigma_\mu = \frac{T}{\sqrt{2}} \sqrt{1 - \left[1 - 2M \left(\frac{dM}{dT} \right) \right]^{-1}}$$

Specific Heat

$$C_V = \frac{1}{2} \left(1 - 2 \left(\frac{\Sigma_\mu}{T} \right)^2 \right)^{-1}$$



Latora, Rapisarda, Ruffo; PRL 80 (1998) 692, Physica D 131 (1999) 38

Kuramoto Model

$$\dot{\theta}_i = \omega_i + \frac{K}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i) \qquad i = 1, \ldots, N \qquad \omega_i = natural\ frequencies$$

In the Kuramoto model, despite the driving forces represented by the (different and time-independent) natural frequencies ω_i of the oscillators, the system exhibits a spontaneous transition from incoherence to collective synchronization as the coupling strength K is increased beyond a threshold value K_c, which depends (as Kuramoto shown in a beautiful analysis) only on the distribution g(ω) of the natural frequencies. The shape of the transition depends on the g(ω) too. Peaks in the Largest Lyapunov Exponent near the phase transitions are very similar to the HMF one (expecially for the Gaussian g(ω)), and indicates again strong chaoticity.

Theoretical critical point

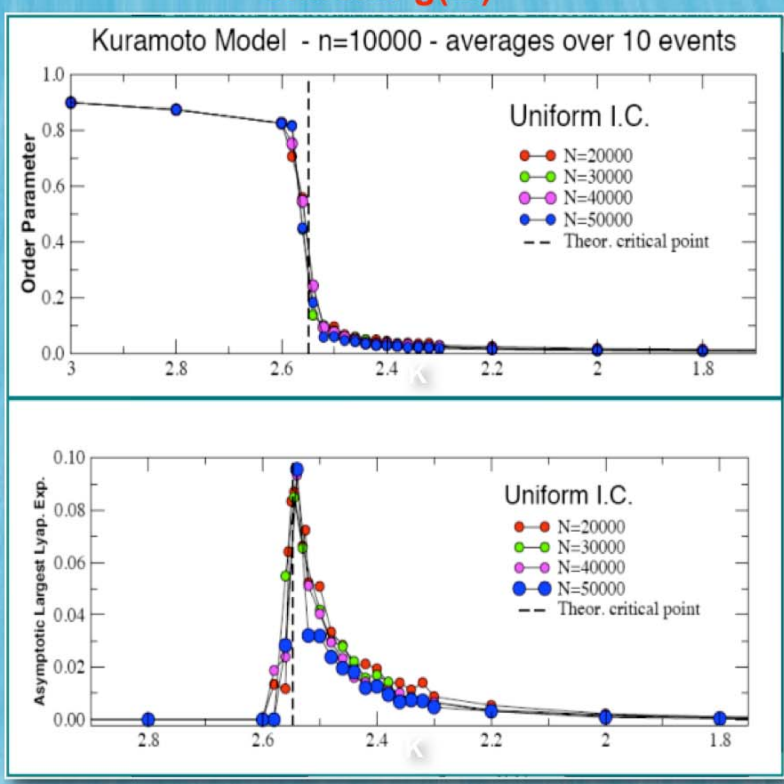
$$K_c = \frac{2}{\pi g(0)}$$

Asymptotic Largest Lyapunov exponent

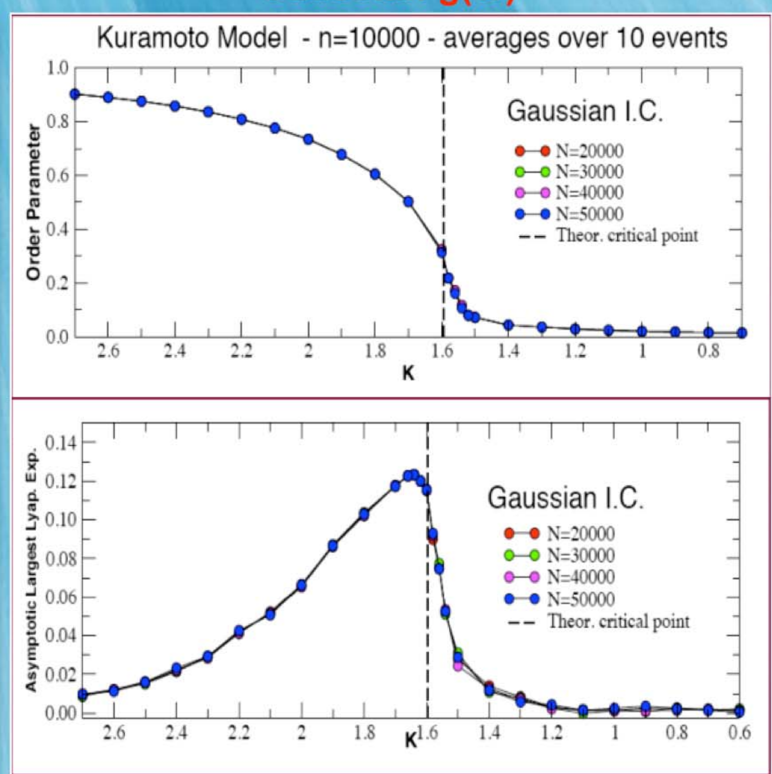
$$\lambda = \lim_{t \to \infty} \frac{1}{t} \ln \frac{d(t)}{d(0)}$$

$$d(t) = \sqrt{\sum_i (d\theta_i)^2}$$

Uniform g(ω)



Gaussian g(ω)

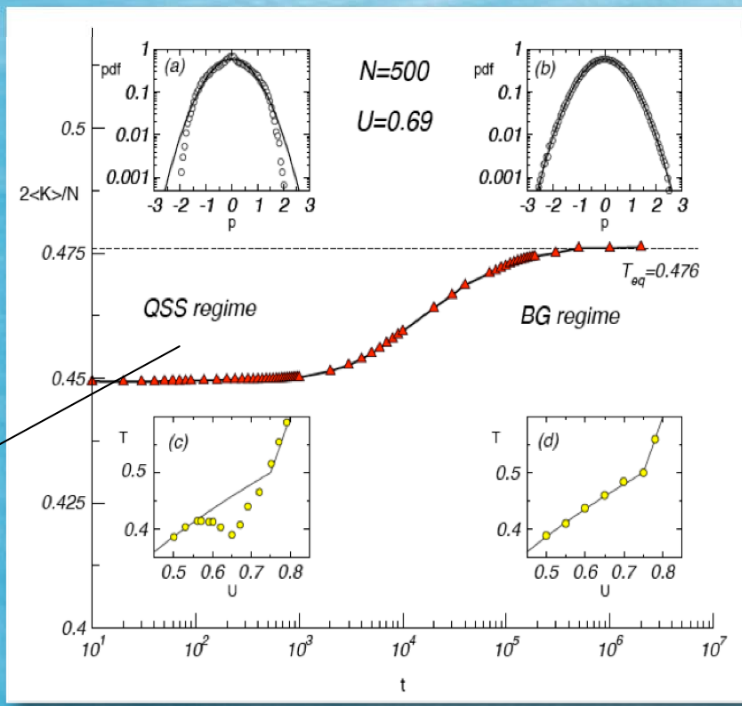


G.Miritello, Master Degree Thesis, unpublished; G.Miritello, A.Pluchino, A.Rapisarda, in preparation

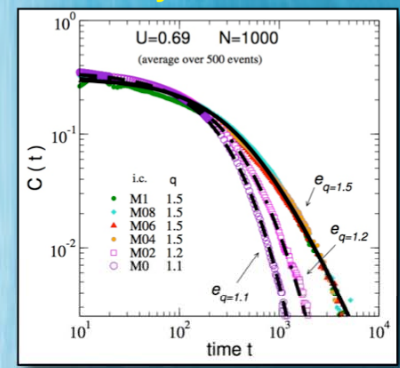
Metastability and Out-of-equilibrium Behavior

HMF Model

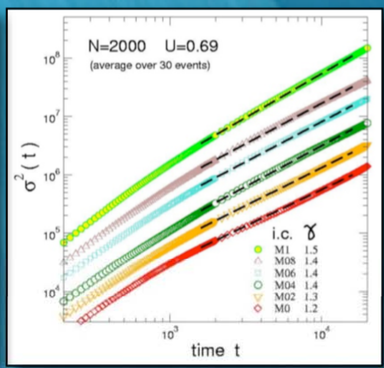
Starting the system from out-of-equilibrium initial conditions (e.g. with all the angles = 0), in the range of energy density [0.5,0.75] one observes a metastable quasistationary (QSS) regime whose lifetime diverges with the system size N. This regime is characterized by non-Gaussian velocity distributions, negative specific heat, fractal structures in the μ-space, weak chaos and vanishing Lyapunov exponents, anomalous diffusion and Levy flights, velocity correlations and aging, glassy dynamics and weak ergodicity breaking. Connections between these anomalies and nonextensive thermostatistics have been widely shown (see refs. below).



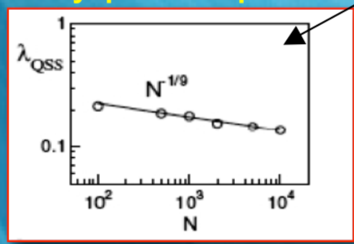
Velocity correlations



Anomalous diffusion



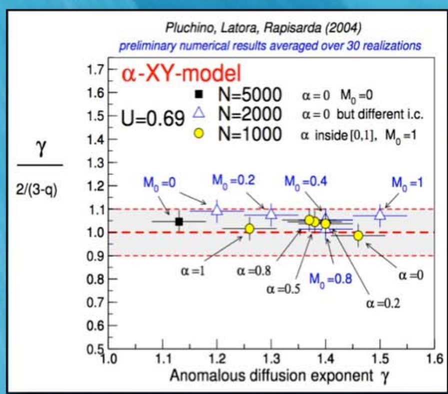
L. Lyapunov Exponent



q-exponential

$$e_q(-x) = A(1-(1-q)x/B)^{\frac{1}{1-q}}$$

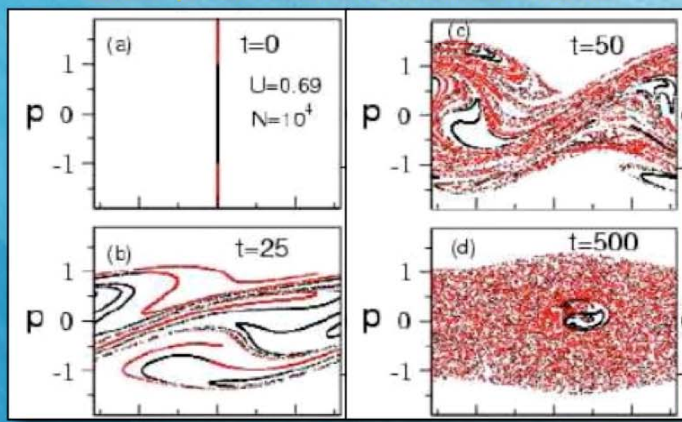
q vs γ Conjecture



A relationship between the entropic index q of velocity correlations decay and anomalous diffusion exponent γ has been suggested for the HMF model and for its generalization (the α-XY model, with variable range of interaction), and seems to be confirmed by many simulations with different initial conditions.

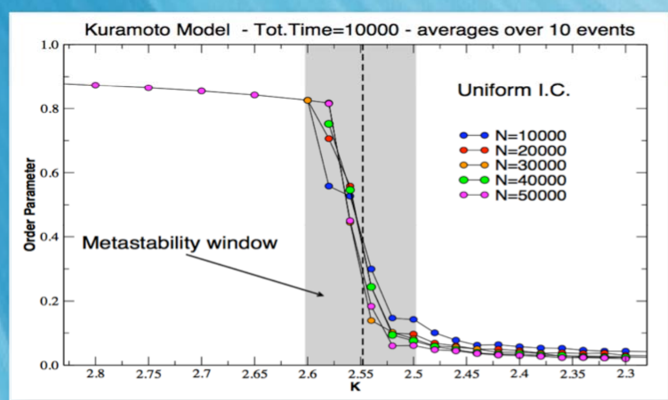
-A.Pluchino, V.Latora, A.Rapisarda, Physica D 193 (2004) 315-328
-A.Pluchino, V.Latora, A.Rapisarda, Physica A 338 (2004) 60-67
-A.Pluchino, V.Latora, A.Rapisarda, Phys.Rev.E 69(2004) 056113
-A.Rapisarda, A.Pluchino, Europhysics News, 36 (2005) 202
-A.Pluchino, A.Rapisarda, Physica A 370 (2006) 573

QSS μ-SPACE EVOLUTION

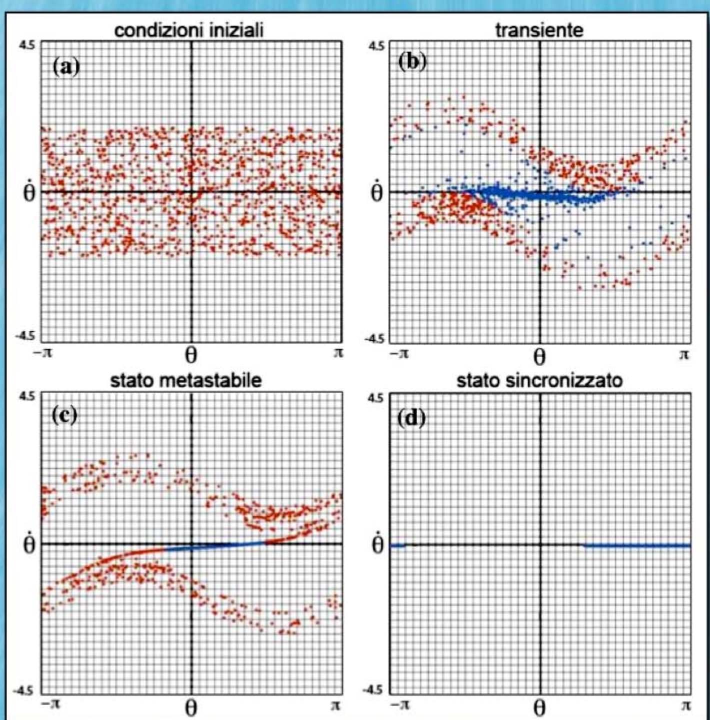


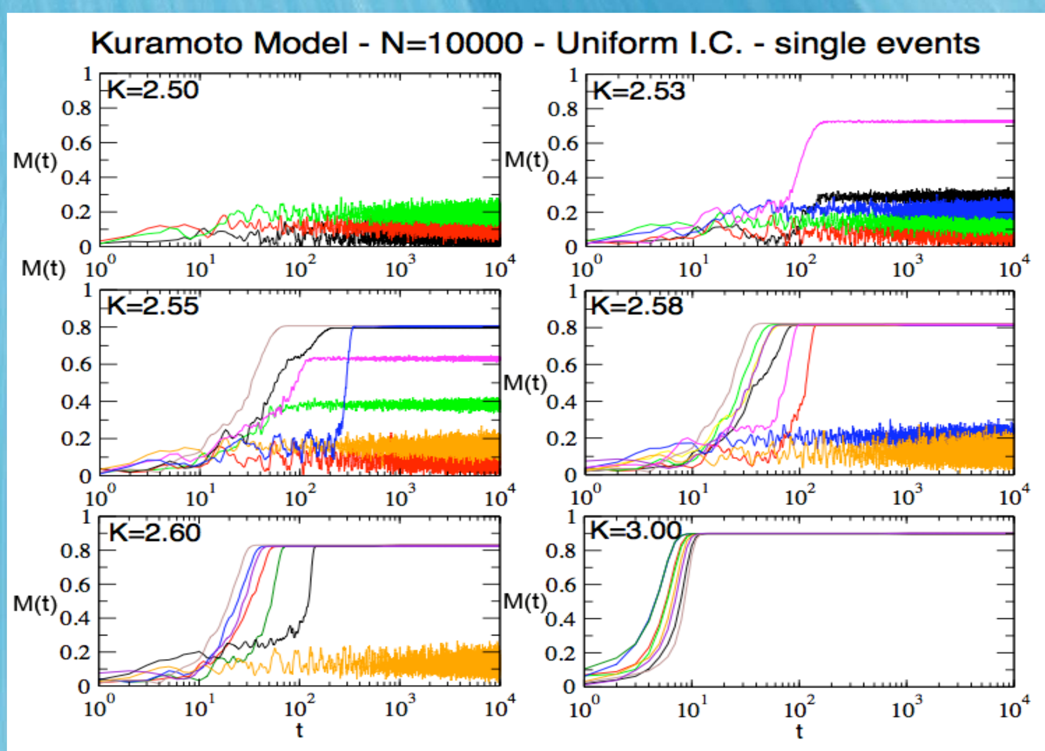
Kuramoto Model

Also in the Kuramoto model a metastable regime has been recently discovered just near the phase transition, starting from homogeneous initial conditions with both uniform and Gaussian g(ω). The metastable regime hinders synchronization due to the presence of two kinds of oscillators, the locked ones (in blue) and the drifting ones (in red), clearly visible in the μ-space, which induce a sort of "dynamical frustration" very similar to that observed in the QSS regime of the HMF model (where clusters compete in attracting the particles).



-A.Pluchino, A.Rapisarda, Physica A 365 (2006) 184
-G.Miritello, Master Degree Thesis, unpublished
-G.Miritello, A.Pluchino, A.Rapisarda, in preparation





Central Limit Theorem and Nonextensive Thermostatistics

The standard Central Limit Theorem (CLT) states that a (conveniently centered and scaled) sum of n→∞ independent (or nearly independent) random variables with finite variance has a Gaussian distribution. Very recently a generalized version of the CLT has been proposed by Tsallis et al. for taking into account those complex systems where long-range correlations are the rule, such as those addressed by nonextensive statistical mechanics. Both HMF and Kuramoto show a behavior that seems to fulfill the requirements of the generalized CLT.

HMF Model

Correlations, weak chaos and non-ergodicity characterizing the QSS regime are so evident that, when one considers Pdfs of sums (y) of the velocities p_K (k=1,…,N) calculated along the deterministic trajectory of each rotor at fixed intervals of time δ, the standard CLT is no longer applicable and q-Gaussian attractors appear.

$$y_k = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} p_k(i\delta) \qquad k = 1,\dots,N$$

$$q-Gaussian: \ p(x) = Ae_q(-\beta x^2)$$

As a further consequence of non-ergodicity, in the QSS regime time averages result to be non equivalent to ensemble averages.

We recently shown that both Gaussian and q-Gaussian attractors coexist in the QSS regime, depending on the particular realization of the same M1 water bag initial conditions. More precisely, three classes of events have been identified, but only for class 1 events asymptotic q-Gaussian attractors appear.

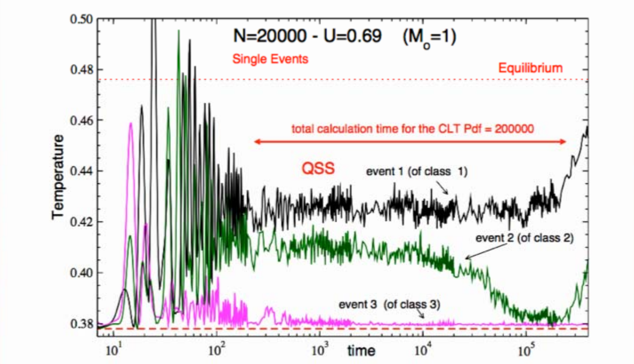
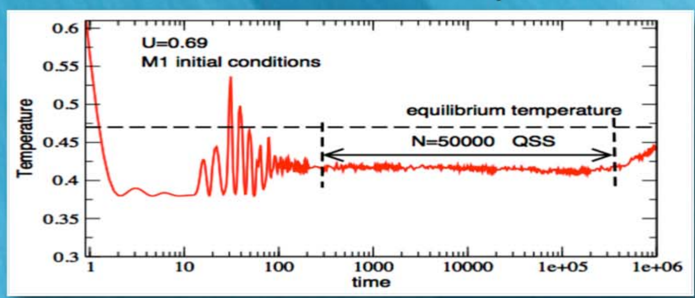
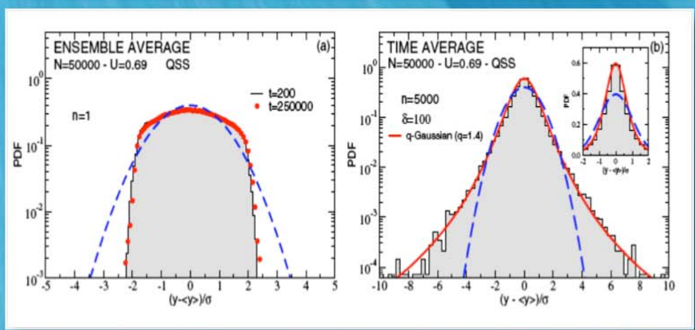


Fig. 1. Time evolution of the temperature (calculated as twice the average kinetic energy per particle) for three single events representative of the three different classes observed at $U = 0.69$ for initial magnetization $M_0 = 1$. The size of the system is $N = 20000$.





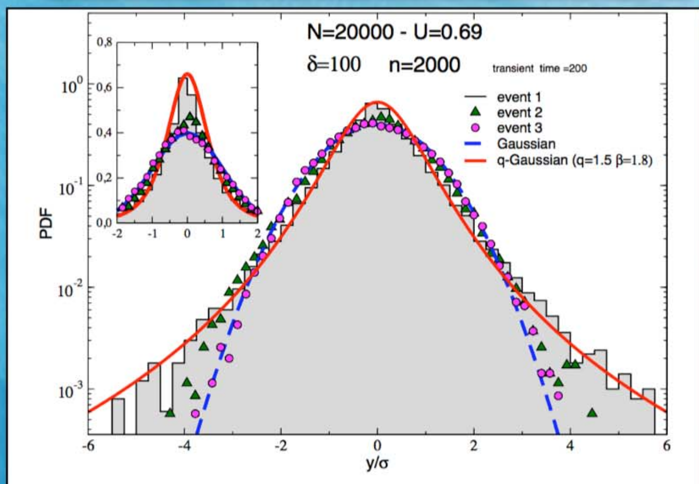


Fig. 3. We present for each class of the QSS found, the different central limit theorem behavior observed. A Gaussian (dashed curve) with unitary variance and a q-Gaussian $p(x) = Ae_q(-\beta x^2)$ with $A = 0.66$ $q = 1.5$ and $\beta = 1.8$ (full curve) are also reported for comparison. In the inset a magnification of the central part in linear scale is plotted. See text for further details.

-A.Pluchino, A.Rapisarda, C.Tsallis, Europhysics Letters 80 (2007) 26002

-A.Pluchino, A.Rapisarda, C.Tsallis, PhysicaA 387 (2008) 3121–3128

-C.Tsallis, A.Rapisarda, A.Pluchino, E.P.Borges, Physica A 341, 143-147 (2007)

Kuramoto Model

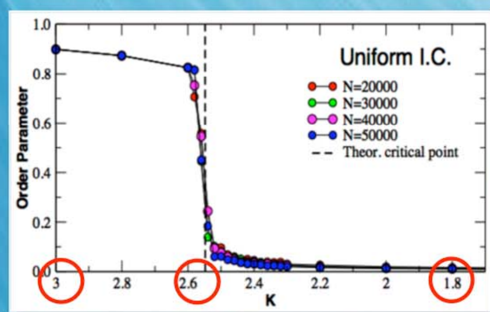
$$y_k = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \theta_k(i\delta) \qquad k = 1,\dots,N$$

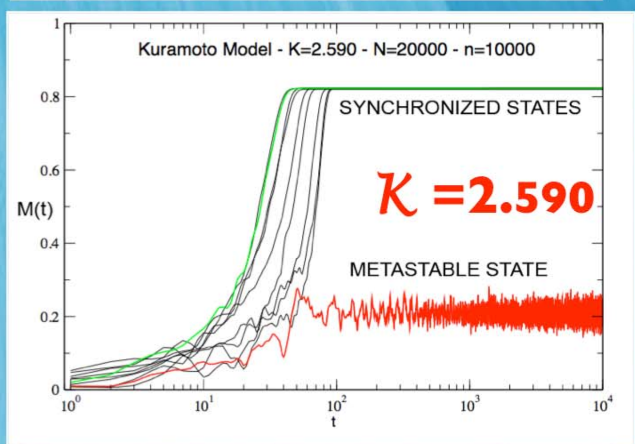
Violation of the standard CLT has been recently observed also in the metastable regime of the Kuramoto model.

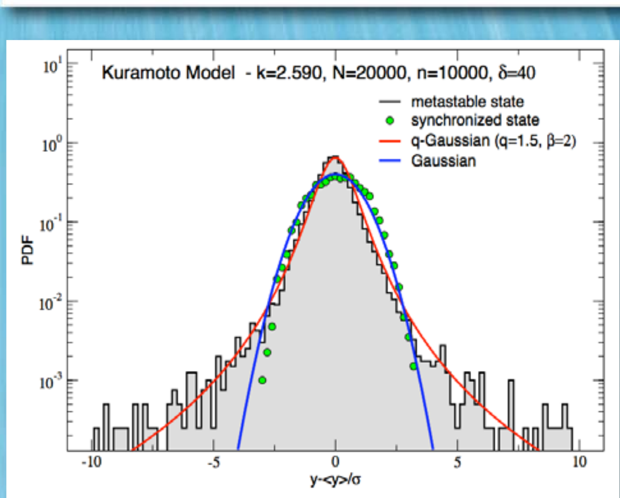
In this case we sum the angles θ_K, calculated during the deterministic trajectory of single oscillators at fixed intervals of time δ and we plot the distribution of these sums y.

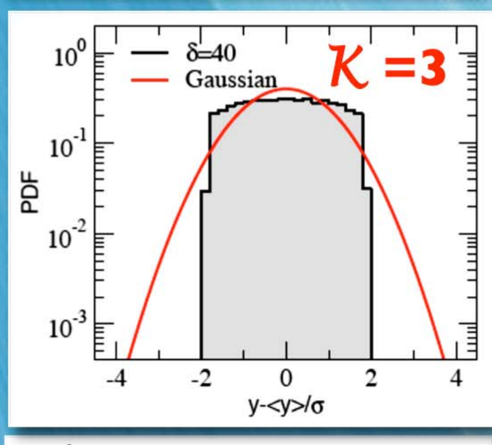
We observe asymptotic q-Gaussian-like attractors along metastable states for K near K_C, while standard Gaussian attractors appear in (partially) synchronized states (Masympt≈0.8) for the same value of K (and for both uniform and Gaussian g(ω) distributions).

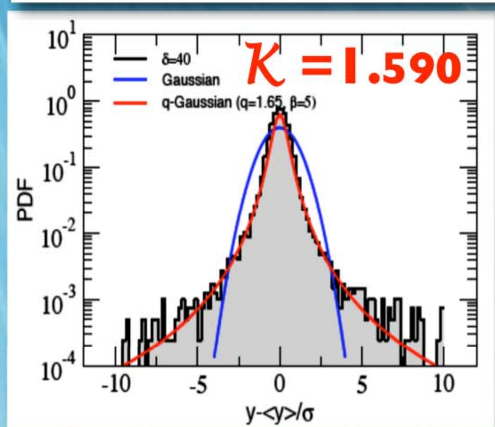
In this context the violation of the CLT seems to be due, again, to the complex interplay between locked and drifting oscillators. A detailed study of this phenomenon, in relation with the Lyapunov Exponents behavior, is in preparation. Uniform Pdfs comes out for high values of the coupling strength (i.e. K=3), since in this case the oscillators are fully synchronized and the system is trivially integrable. On the other hand, q-Gaussian attractors have been observed also in the incoherent phase (i.e. for small values of K), even if in this case their origin is still unclear and is actually object of a deeper exploration.











-G.Miritello, Master Degree Thesis, unpublished

-G.Miritello, A.Pluchino, A.Rapisarda, in preparation